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MIXED PROBLEMS OF THE MECHANICS OF CONTINUOUS MEDIA ASSOCIATED WITH HANKEL AND MEHLER-FOCK INTEGRAL TRANSFORMS

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On the basis of the Hankel and Mehler-Fock integral transforms we consider some types of dual integral equations and the corresponding types of integral equations of the second kind which occur in the study of a series of mixed problems of the mechanics of continuous media. By generalizing the asymptotic methods of [1, 2], effective approximate solutions are obtained. As an example we investigate the torsion, induced by a punch, of a truncated sphere, fixed along the spherical boundary.

1. Problems associated with the Hankel transform. We consider the dual integral equation

$$\int_0^\infty T(\gamma) J_n(\gamma r) L(\lambda\gamma)\, d\gamma = f(r) \qquad (0 \leqslant r \leqslant 1)$$

$$\int_0^\infty T(\gamma) J_n(\gamma r) \gamma\, d\gamma = 0 \qquad (r > 1) \tag{1.1}$$

Here $0 < \lambda < \infty$ is a dimensionless parameter, $J_n(x)$ are the Bessel functions, the continuous function $L(u) > 0$ for $u > 0$ is such that

$$L(u) = Bu(1 + O(u^2)) \qquad (u \to 0,\ B = \text{const}) \tag{1.2}$$

$$L(u) = 1 - \sum_{i=1}^{m} c_i u^{-2i} + O(u^{-2(m+1)}) \qquad (u \to \infty)$$

Let $n = 1$. We multiply the first relation of (1.1) by $(t^2 - r^2)^{-1/2}$ and we integrate with respect to r from zero to t and we multiply the second relation by $(r^2 - t^2)^{-1/2}$ and we integrate with respect to r from t to infinity. Interchanging the order of integration, making use of the integrals

$$\int_0^t \frac{J_1(\gamma r)\, dr}{\sqrt{t^2 - r^2}} = \frac{1 - \cos\gamma t}{\gamma t}, \qquad \int_t^\infty \frac{J_1(\gamma r)\, dr}{\sqrt{r^2 - t^2}} = \frac{\sin\gamma t}{\gamma t} \tag{1.3}$$

and differentiating then the first equation with respect to t, we obtain

$$\varphi(t) = \int_0^\infty T(\gamma)[1 - L(\lambda\gamma)] \sin\gamma t\, d\gamma = g'(t) \qquad (0 \leqslant t \leqslant 1) \tag{1.4}$$

$$\varphi(t) = 0 \qquad (t > 1)$$

$$g(t) = \int_0^t \frac{f(r)\, dr}{\sqrt{t^2 - r^2}}, \qquad \varphi(t) = \int_0^\infty T(\gamma) \sin\gamma t\, d\gamma \tag{1.5}$$

With regard to the second equality of (1.4)

$$T(\gamma)=\frac{2}{\pi}\int_0^1\varphi(\tau)\sin\gamma\tau\,d\tau \tag{1.6}$$

Substituting (1.6) into the first equality of (1.4), we arrive at the integral equation of the second kind with respect to $\varphi(t)$

$$\varphi(t)=\frac{2}{\pi}\int_0^1\varphi(\tau)\,d\tau\int_0^\infty[1-L(\lambda\gamma)]\sin\gamma t\sin\gamma\tau\,d\gamma+g'(t)\qquad(0\leqslant t\leqslant 1) \tag{1.7}$$

Assuming that $\varphi(t)$ and $g'(t)$ are odd functions, we obtain the equation

$$\varphi(t)=\frac{1}{\pi\lambda}\int_{-1}^1\varphi(\tau)M\left(\frac{t-\tau}{\lambda}\right)d\tau+g'(t)\qquad(|t|\leqslant 1) \tag{1.8}$$

$$M(y)=\int_0^\infty[1-L(u)]\cos uy\,du \tag{1.9}$$

Relations (1.8), (1.9) can be written also in the form

$$\int_{-1}^1\varphi(\tau)K\left(\frac{t-\tau}{\lambda}\right)d\tau=\pi\lambda g'(t)\qquad(|t|\leqslant 1) \tag{1.10}$$

$$K(y)=\int_0^\infty L(u)\cos uy\,du \tag{1.11}$$

In specific problems we frequently make use of the quantity

$$\tau(r)=\int_0^\infty T(\gamma)J_1(\gamma r)\gamma\,d\gamma \tag{1.12}$$

which can be expressed in terms of $\varphi(t)$ as follows :

$$\tau(r)=-\frac{2}{\pi}\frac{d}{dr}\int_r^1\frac{\varphi(\tau)\,d\tau}{\sqrt{\tau^2-r^2}}\qquad(0\leqslant r\leqslant 1) \tag{1.13}$$

For the case $n=0$ we can arrive in a similar manner to the equations (1.8),(1.9) or (1.10), (1.11). Here we take into account that the functions $\varphi(t)$ and $g'(t)$ must be extended into the domain $-1\leqslant t<0$ in an even form and

$$\tau(r)=-\frac{2}{\pi}\frac{d}{dr}r\int_r^1\frac{\varphi(\tau)\,d\tau}{\sqrt{\tau^2-r^2}},\qquad g(t)=\int_0^t\frac{rf(r)\,dr}{\sqrt{t^2-r^2}} \tag{1.14}$$

For $n\geqslant 2$ Eq. (1.1) can also be reduced [3] to an equation of type (1.8), (1.9).

1. We assume that the parameter λ is large and the variable y in (1.9) is small and we can obtain the expansion

$$M(y)=\sum_{i=0}^\infty b_i|y|^i,\qquad b_{2n+1}=\frac{(-1)^{n+1}\pi}{2(2n+1)!}c_{n+1} \tag{1.15}$$

$$b_{2n} = \frac{(-1)^n}{(2n)!} \int_0^\infty \left[1 - L(u) - \sum_{i=1}^{n} \frac{c_i}{u^{2i}}\right] u^{2n}\, du$$

Substituting (1.15) into (1.8) and seeking the solution in the form

$$\varphi(t) = \sum_{k=0}^{\infty} \varphi_k(t) \lambda^{-k} \tag{1.16}$$

we obtain for $\varphi_k(t)$ the following recursion relation: $\varphi_0(t) = g'(t)$

$$\varphi_k(t) = \frac{1}{\pi} \sum_{i=0}^{k-1} b_{k-i-1} \int_{-1}^{1} \varphi_i(\tau) \, |t - \tau|^{k-i+1} \, d\tau \tag{1.17}$$

2. We give another method of solution of (1.8) in the case of sufficiently large λ. For this we expand the kernel $M(y)$ in a double series in Legendre polynomials

$$M\left(\frac{\tau - t}{\lambda}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}(\lambda) P_i(\tau) P_j(t)$$

$$c_{nm} = (-1)^{n+m} (2n+1)(2m+1) \frac{\pi\lambda}{2} \int_0^\infty [1 - L(u)] \times$$

$$\times J_{1/2+n}\left(\frac{u}{\lambda}\right) J_{1/2+m}\left(\frac{u}{\lambda}\right) \frac{du}{u} \tag{1.18}$$

Substituting (1.18) into (1.8) and, in the case of an even function $g'(t)$, seeking the solution in the form

$$\varphi(t) = \sum_{k=0}^{\infty} S_k P_{2k}(t) \tag{1.19}$$

we obtain with respect to the unknown coefficients S_k the following infinite system:

$$S_i = \frac{2}{\pi\lambda} \sum_{k=0}^{\infty} S_k c_{2i,\,2k}(\lambda) \, (4k+1)^{-1} + G_i \quad (i = 0, 1, \ldots) \tag{1.20}$$

Here G_i are the coefficients of the expansion of the function $g'(t)$ in a series of the form (1.19). In a similar way, an infinite system in the case of an odd function $g'(t)$. can be obtained.

3. Assume that the parameter λ is small. We approximate the function $L(u)$ with the properties (1.2) by the expression

$$L^*(u) = \frac{u}{\sqrt{u^2+1}} \frac{Q_1(u)}{Q_2(u)} \tag{1.21}$$

where $Q_1(u)$ and $Q_2(u)$ are even polynomials of the same degree. We consider in detail the simplest case when $Q_1 = Q_2 = 1$. In the case $g'(t) \equiv 1$ we obtain the approximate solution of Eqs. (1.10), (1.11) from the formula

$$\varphi(t) = \lim q_\varepsilon(t, s) \qquad \left(\begin{matrix} \varepsilon \to 0 \\ s = 1 \end{matrix}\right) \tag{1.22}$$

where the function $q_\varepsilon(t, s)$ is obtained from the integral equation

$$\int_{-s}^{s} q_\varepsilon(\tau, s)\,d\tau \int_0^\infty \frac{\sqrt{u^2+\varepsilon^2}}{\sqrt{u^2+1}} \cos\left(\frac{\tau-t}{\lambda}u\right) du = \pi\lambda \qquad (|t| \leqslant s \leqslant 1) \tag{1.23}$$

We will seek the principal terms of the asymptotics for small λ of the solution of Eq. (1.23) in the form

$$q_\varepsilon{}^\circ(t, s) = \mu\omega\left(\frac{s+t}{\lambda}\right)\omega\left(\frac{s-t}{\lambda}\right)v^{-1}\left(\frac{t}{\lambda}\right) \tag{1.24}$$

where $\omega(\tau)$ and $v(\tau)$ are the solutions of the equations

$$\int_0^\infty \omega(\tau)\,d\tau \int_{-\infty}^{\infty} \frac{\sqrt{u^2+\varepsilon^2}}{\sqrt{u^2+1}} e^{-i(t-\tau)u} du = 2\pi \qquad (0 \leqslant t < \infty) \tag{1.25}$$

$$\int_{-\infty}^\infty v(\tau)\,d\tau \int_{-\infty}^{\infty} \frac{\sqrt{u^2+\varepsilon^2}}{\sqrt{u^2+1}} e^{-i(t-\tau)u} du = 2\pi \qquad (|t| < \infty) \tag{1.26}$$

Omitting the computations, we give the result

$$\omega(t) = \frac{1}{\sqrt{\varepsilon}}\left[\exp\left(-\frac{\varepsilon+1}{2}t\right) I_0\left(\frac{1-\varepsilon}{2}t\right) + \right. \tag{1.27}$$

$$\left. + \int_0^t \exp\left(-\frac{\varepsilon+1}{2}\tau\right) I_0\left(\frac{\varepsilon-1}{2}\tau\right) d\tau\right], \qquad v(t) = \frac{1}{\varepsilon}$$

Here $I_0(x)$ is the modified Bessel function. Now according to (1.22) and (1.24) we obtain

$$\varphi_0(t) = q_0(t, 1) \qquad q_0(t, s) = \mu F\left(\frac{s+t}{\lambda}\right) F\left(\frac{s-t}{\lambda}\right) \tag{1.28}$$

$$F(x) = \Phi(-{}^1/_2,\ 1;\ -x)$$

where $\Phi(\alpha, \beta; x)$ is the confluent hypergeometric function. Substituting (1.28) into Eqs. (1.10), (1.21) for $g'(t) \equiv 1$, $Q_1 = Q_2 \equiv 1$ and passing to the limit for $\lambda \to 0$, we obtain that $\mu = \pi / 4$. Now, from the formulas (1.28) we obtain

$$\varphi_0(0) = \left(\frac{1}{\lambda} + 1 + O(\lambda)\right), \qquad \varphi_0(1) = \sqrt{\frac{\pi}{2\lambda}}(1 + O(\lambda)) \tag{1.29}$$

The same formulas can be obtained by making use of the results of [4]. This confirms once again that the relation (1.28) gives the principal term of the asymptotics of the solution for small λ.

4. If, as before, we seek the principal term of the asymptotics by the formulas (1.22) (1.24), while the functions $\omega(t)$ and $v(t)$ are determined from the equations

$$\int_0^\infty \omega(\tau)\,d\tau \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{u^2+1}} e^{-i(t-\tau)u} du = 2\pi e^{-\varepsilon t} \qquad (0 \leqslant t < \infty)$$

$$\int_{-\infty}^\infty v(\tau)\,d\tau \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{u^2+1}} e^{-i(t-\tau)u} du = 2\pi e^{-\varepsilon t} \qquad (|t| < \infty) \tag{1.30}$$

then we obtain formulas (1.28). Indeed,

$$\omega(t) = \frac{\sqrt{\varepsilon+1}}{\sqrt{\pi\varepsilon(1-\varepsilon)}}\left(\frac{d}{dt} l(t, \varepsilon) + l(t, \varepsilon)\right), \qquad v(t) = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon} e^{-\varepsilon t}$$

$$l(t, \varepsilon) = \int_0^t \frac{e^{-\varepsilon\tau}}{\sqrt{t-\tau}} \operatorname{erf} \sqrt{(1-\varepsilon)\tau}\, d\tau \tag{1.31}$$

Here erf x is the probability integral. In the derivation of the formulas (1.31) we have made use of the following relations (*)

$$|u| = -[\sqrt{u}]^+ [\sqrt{u}]^-$$

$$[\sqrt{u}]^+ = \begin{cases} \sqrt{u} & (u \geqslant 0) \\ i\sqrt{u} & (u < 0) \end{cases}, \quad [\sqrt{u}]^- = \begin{cases} -\sqrt{u} & (u \geqslant 0) \\ i\,|\sqrt{u}| & (u < 0) \end{cases} \tag{1.32}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-iut}\, du}{\sqrt{u+i}\,(u+i\varepsilon)\,[\sqrt{u}]^+} = \begin{cases} -\sqrt{2}\,(1-\varepsilon)^{-1/2}\, l(t,\varepsilon) & (t \geqslant 0) \\ 0 & (t < 0) \end{cases}$$

5. If the right-hand side of Eq. (1.10) is not a constant, then for small λ the asymptotic solution can be obtained from Krein's formula [5] by making use of (1.28).

Let $g'(t) = At$, $A = \text{const}$. Then

$$\varphi_0(t) = -\frac{A}{2}\frac{d}{dt} \int_{|t|}^{1} \frac{q_0(t, \xi)}{q_0^2(\xi, \xi)} \left(\int_{-\xi}^{\xi} q_0(\tau, \xi)\, d\tau \right) d\xi \tag{1.33}$$

Making use of the integral

$$\int_{-\xi}^{\xi} q_0(\tau, \xi)\, d\tau = \frac{\pi}{2}\left(\xi + \frac{\xi^2}{\lambda}\right) \tag{1.34}$$

and simplifying (1.33) asymptotically, we obtain

$$\varphi_0(t) = A\left[F\left(\frac{2t}{\lambda}\right)\right]^{-1}\left(t + \frac{t^2}{\lambda}\right) + \frac{At}{2\lambda}(1-t^2)^{1/2}(1 + O(\lambda)) \tag{1.35}$$

If $g'(t) = At^2$, $A = \text{const}$, then in a similar manner we can obtain

$$\varphi_0(0) = \frac{A}{6\lambda}(1 + O(\lambda)), \qquad \varphi_0(1) = \frac{A}{2}\sqrt{\frac{\pi}{2\lambda}}(1 + O(\lambda)) \tag{1.36}$$

6. We give some computational results which allow to establish the given scheme for obtaining the approximate solution will ensure the complete and efficient investigation of one or another problem for all values of the parameter λ.

For the sake of simplicity we consider the case when $g'(t) \equiv 1$ and $L(u) = u(u^2+1)^{-1/2}$. Then the constants c_i in (1.2) and b_i in (1.15) have the form

$$c_i = (-1)^{i-1}\frac{(2i-1)!!}{(2i)!!} \tag{1.37}$$

$$b_{2n} = \frac{(-1)^n}{(2n)!\,2^{2n}} \sum_{i=0}^{n} \frac{(-1)^i (2n+1)!}{i!\,(2n-2i+1)\,(2n-i+1)!}$$

$$b_{2n+1} = -\frac{\pi}{[(2n)!!]^2\,(4n+4)} \tag{1.38}$$

According to (1.17) for large λ we obtain

*) The formulas (1.32) are taken from the Iu. I. Cherskii doctoral dissertation, Tbilisi, 1962.

$$\varphi_0(t) = 1, \quad \varphi_1(t) = 2/\pi; \quad \varphi_2(t) = \eta_1 t^2 + \eta_2$$
$$\varphi_3(t) = \eta_3 t^2 + \eta_4, \quad \varphi_4(t) = \eta_5 t^4 + \eta_6 t^2 + \eta_7 \tag{1.39}$$
$$\varphi_5(t) = \eta_8 t^4 + \eta_9 t^2 + \eta_{10}$$

$\eta_1 = -0.2500$	$\eta_5 = -0.004987$	$\eta_8 = 0.001989$
$\eta_2 = 0.1553$	$\eta_6 = 0.002520$	$\eta_9 = -0.02945$
$\eta_3 = 0.05305$	$\eta_7 = 0.005968$	$\eta_{10} = 0.0009780$
$\eta_4 = -0.04261$		

The first three coefficients (1.18) have the form

$$c_{00} = 2\sum_{k=0}^{\infty}\frac{(2/\lambda)^{2k}}{[(2k+1)!!]^2(2k+2)} - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(1/\lambda)^{2k-1}}{(k!)^2(2k+1)} \tag{1.40}$$

$$c_{02} = c_{20} = 4\sum_{k=0}^{\infty}\frac{(2/\lambda)^{2k}[30-15(2k+4)+2.5(2k+3)(2k+4)]}{[(2k+1)!!]^2(2k+2)(2k+3)(2k+4)} -$$
$$-\pi\sum_{k=1}^{\infty}\frac{(1/\lambda)^{2k-1}[30-15(2k+3)+2.5(2k+2)(2k+3)]}{(k!)^2(2k+1)(2k+2)(2k+3)}$$

We solve system (1.13) by the method of reduction, setting $i + k \leqslant 1$. We have, respectively

$\lambda = 0.4$	0.6	0.8	1.0	1.2	1.4	
$S_0 = 2.80$	2.18	1.87	1.69	1.57	1.48	(1.41)
$-S_1 = 0.514$	0.293	0.187	0.129	0.0937	0.0712	

In Table 1 the values of

$$\varphi(0), \quad \varphi(1), \quad P = \int_{-1}^{1}\varphi(t)\,dt \tag{1.42}$$

are given, computed with the formulas (1.39) (lines 2 – 4), with the formulas (1.19), (1.41) (lines 5 – 7) and with the formulas (1.28), (1.34) (lines 8 – 10).

2. Problems associated with the Mehler-Fock transform. We consider the dual integral equations

$$\int_0^{\infty} T(\gamma)P^n_{-1/2+i\gamma}(\operatorname{ch} r)L(\gamma)\,d\gamma = f(r) \qquad (0 \leqslant r \leqslant \alpha) \tag{2.1}$$
$$\int_0^{\infty} T(\gamma)P^n_{-1/2+i\gamma}(\operatorname{ch} r)\gamma \operatorname{th}\pi\gamma\,d\gamma = 0 \qquad (r > \alpha)$$

where $0 < \alpha < \infty$ is a dimensionless parameter, $P^n_{-1/2+i\gamma}$ are the associated conical functions, the continuous function $L(\gamma) > 0$ for $\gamma > 0$ is such that

$$L(\gamma) = B\gamma^2(1 + O(\gamma^2)) \qquad (\gamma \to 0, \ B = \text{const}) \tag{2.2}$$
$$L(\gamma) = 1 + O(\gamma^{-2}) \qquad (\gamma \to \infty)$$

The dual integral equations (2.1) can be reduced to the integral equation of the second kind [3]

$$\varphi(x) = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\varphi(t)M(t-x)\,dt + p(x) \qquad (|x| \leqslant \alpha)$$

$$M(y) = \int_0^{\infty}(1 - L(\gamma))\cos\gamma y\,d\gamma \tag{2.3}$$

The relations (2.3) can also be written in the form

$$\int_{-\alpha}^{\alpha} \varphi(t) K(t-x)\, dt = \pi p(x) \qquad (|x| \leqslant \alpha) \tag{2.4}$$

$$K(y) = \int_0^\infty L(\gamma) \cos \gamma y d\gamma \tag{2.5}$$

Here for the case $n = 1$

$$T(\gamma) = \int_0^\alpha \varphi(t) \cos \gamma t dt$$

$$p(x) = c \operatorname{ch} \frac{x}{2} + \frac{\sqrt{2} \operatorname{sh} x}{\pi} \int_0^x \frac{f(\tau)\, d\tau}{\sqrt{\operatorname{ch} x - \operatorname{ch} \tau}} \tag{2.6}$$

In the sequel the quantity

$$\psi(r) = \int_0^\infty T(\gamma) \gamma \operatorname{th} \pi\gamma\, P^1_{-1/2+i\gamma}(\operatorname{ch} r)\, d\gamma \tag{2.7}$$

will be necessary which, by taking into account (2.6), can be given the form [3]

$$\psi(r) = -\frac{d}{dr} \int_r^\alpha \varphi'(\tau) [2(\operatorname{ch} \tau - \operatorname{ch} r)]^{-1/2} d\tau \qquad (0 \leqslant r \leqslant \alpha) \tag{2.8}$$

Here we consider $\varphi(\alpha) = 0$ which corresponds to the condition of integrability of $\psi(r)$ on $[0, \alpha]$ and serves as a condition for the determination of the constant c in (2.6).

We note that the dual integral equations

$$\int_0^\infty \gamma T(\gamma) P^n_{-1/2+i\gamma}(\operatorname{ch} r) L(\gamma)\, d\gamma = f(r) \qquad (0 \leqslant r \leqslant \alpha)$$

$$\int_0^\infty T(\gamma) \operatorname{th} \pi\gamma\, P^n_{-1/2+i\gamma}(\operatorname{ch} r)\, d\gamma = 0 \qquad (r > \alpha) \tag{2.9}$$

can also be reduced to an equation of type (2.3) [3].

The approximate solution of the equations (2.4), (2.5) can be constructed by approximating the function $L(\gamma)$ in accordance with (2.2) by the expression

$$L^*(\gamma) = \frac{\gamma^2 P_1(\gamma)}{(\gamma^2 + D^2) P_2(\gamma)} \tag{2.10}$$

where $P_1(\gamma)$ and $P_2(\gamma)$ are even polynomials of the same degree. Further, we examine in detail the case (a) $P_1 = P_2 \equiv 1$ and touch upon the case

b) $P_1(\gamma) = \gamma^2 + E^2, \qquad P_2(\gamma) = \gamma^2 + G^2.$

The solution of the integral equations (2.4), (2.5), (2.10) can be obtained in the closed form [2]. Taking into account Krein's formulas [5] we give, omitting the computations, the solution of the equation

$$\int_{-s}^{s} q(t, s) K(t - x)\, dt = \pi \qquad (|x| \leqslant s \leqslant \alpha) \tag{2.11}$$

For the case (a) we have

$$q(t, s) = {}^1/_2 D^2 s^2 + Ds + 1 - {}^1/_2 D^2 t^2 \tag{2.12}$$

For the case (b) the solution is considerably more complicated

$$q(t, s) = m - nt^2 + l\,[e^{-E(s+t)} + e^{-E(s-t)}]$$

$$m = [\Phi_1 s^2 + \Phi_2 s + \Phi_3 + e^{-2Es}(\Phi_4 s^2 + \Phi_5 s + \Phi_6)]\,[E_1 + E_2 e^{-2Es}]^{-1} \tag{2.13}$$

$$n = D^2 G^2 2^{-1} E^{-2} = (2\pi\beta)^{-1}, \qquad l = (F_1 + F_2 s)\,(E_1 + E_2 e^{-2Es})^{-1}$$

$$E_1 = E\,(D - G) D^{-1} G^{-1}\,(E - D)^{-1}\,(G - E)^{-1}$$

$$E_2 = E\,(D - G) D^{-1} G^{-1}\,(D + E)^{-1}\,(G + E)^{-1}$$

$$\Phi_1 = DG\,(G - D) 2^{-1} E^{-1}\,(D - E)^{-1}\,(G - E)^{-1}$$

$$\Phi_2 = [D^2\,(G - E)] - G^2\,(D - E)]\,E^{-2}\,(D - E)^{-1}\,(G - E)^{-1}$$

$$\Phi_3 = [D^3\,(G - E) - G^3\,(D - E)[E^{-2} G^{-1} D^{-1}\,(D - E)^{-1}\,(G - E)^{-1}$$

$$\Phi_4 = DG\,(D - G)\,2^{-1} E^{-1}\,(D + E)^{-1}\,(G + E)^{-1}$$

$$\Phi_5 = [D^2\,(G + E) - G^2\,(D + E)] E^{-2}\,(D + E)^{-1}\,(G + E)^{-1}$$

$$\Phi_6 = [D^3\,(G + E) - G^3\,(D + E)] E^{-2} G^{-1} D^{-1}\,(D - E)^{-1}\,(G - E)^{-1}$$

$$F_1 = (G^2 - D^2) E^{-2} D^{-1} G^{-1}, \qquad F_2 = (G - D) E^{-2}$$

Assuming that the function $p(x)$ is even and $p(x) = p_1'(x)$, we obtain for the case (a) from Krein's formula [5] and using (2.12) the following approximate solution of the integral equation (2.4), (2.5):

$$\varphi(x) = p(x) + Dp'(\alpha) + D^2 \int_{|x|}^{\alpha} p_1(t)\,dt \tag{2.14}$$

We note that if we make use of the method of construction of the principal term of the asymptotics for small $\lambda = s^{-1}$, given in Sect. 1, Subsection 3, then for Eq. (2.11) in the case (a) we have

$$q(t, s) = {}^1/_2\, D^2 s^2 + Ds + {}^1/_2 - {}^1/_2\, D^2 t^2 \tag{2.15}$$

which coincides exactly with (2.12). For the approximate solution of the equations (2.4) (2.5), (2.2) we can make use of the algorithms of Sect. 1 (see Subsections 1 and 2).

3. The torsion of a truncated sphere by a punch. This problem has been considered earlier in [3, 6], where it has been reduced to dual integral equations of the type (2.1), and then to an equation of the second kind of type (2.3), where

$$L(\gamma) = \operatorname{th} \pi\gamma \operatorname{th} \beta\gamma \quad (0 \leqslant \beta \leqslant \pi)$$

$$\alpha = 2 \operatorname{Arth}(b / a), \qquad \beta = \arcsin\ (a / R) \tag{3.1}$$

$$p(x) = c \operatorname{ch} x / 2 + H (\operatorname{ch} x / 2)^{-1},$$

$$H = -2\sqrt{2}\, a\varepsilon\pi^{-1}$$

Here β is a parameter which characterizes the degree of the truncation of the sphere, a is the radius of the cut, R is the radius of the sphere, b is the radius of the punch, and ε is the rotation angle of the punch.

Obviously, $L(\gamma)$ of the form (3.1) satisfies properties (2.2) for $B = \pi\gamma$. The contact shearing stresses are obtained from the formula [3]

$$\tau(r) = -Ga^{-1}(1 + \operatorname{ch} r)^{3/2}\psi(r), \qquad r = 2 \operatorname{Arth}(\rho a^{-1}) \tag{3.2}$$

where the function $\psi(r)$ is given by (2.8) and ρ is the distance to the axis of symmetry.

As an approximate solution of the equation (2.3), (3.1) we take (2.14) where

$$D = (\pi\beta)^{-1/2}$$

$$p_1(x) = 2c \operatorname{sh}(x/2) + 2H \operatorname{arctg}(sh(x/2)) \tag{3.3}$$

We note that the accuracy of the approximation (2.10) for the simplest case ($P_1 = P_2 \equiv 1$) and for $\pi/4 \leqslant \beta \leqslant \pi$ does not exceed 15%.

Now from the formulas (2.8) and (3.2) we find the approximate representation for $\tau(r)$

$$\begin{aligned}\tau(r) = a^{-1}G \operatorname{sh} r (1 + \operatorname{ch} r)^{3/2} \Big[-\frac{1}{2} c \Big(\frac{1}{2} - 2D^2\Big)\Big((1 + \operatorname{ch} r)^{-1} + \\ + \frac{\sqrt{1+\operatorname{ch}\alpha} + \sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}}{\sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}\,[2\sqrt{(\operatorname{ch}\alpha - \operatorname{ch} r)(1+\operatorname{ch}\alpha)} + 2\operatorname{ch}\alpha - \operatorname{ch} r + 1}\Big) + \\ + H\sqrt{2}D^2\Big(\frac{\operatorname{arc\,tg}(\operatorname{sh} {}^1/_2\alpha)}{\operatorname{sh}\alpha\sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}} + \frac{\sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}}{\sqrt{2}(1+\operatorname{ch} r)\sqrt{\operatorname{ch}\alpha + 1}} - \\ - \frac{1}{4\sqrt{\operatorname{ch} r - 1}} \arccos \frac{4(1-\operatorname{ch} r) + (3 - \operatorname{ch} r)(\operatorname{ch}\alpha - 1)}{(1+\operatorname{ch} r)(\operatorname{ch}\alpha - 1)} + \\ + \int_r^\alpha \frac{\operatorname{ch} x \operatorname{arc\,tg}(\operatorname{sh} {}^1/_2 x)\,dx}{\operatorname{sh}^2 x\sqrt{\operatorname{ch} x - \operatorname{ch} r}}\Big)\Big]\end{aligned} \tag{3.4}$$

where the constant c, obtained from the condition $\varphi(\alpha) = 0$, has the form

$$c = -H\,\frac{(\operatorname{ch} {}^1/_2\alpha)^{-1} + 2D \operatorname{arc\,tg}(\operatorname{sh} {}^1/_2\alpha)}{\operatorname{ch} {}^1/_2\alpha + 2D\operatorname{sh} {}^1/_2\alpha} \tag{3.5}$$

For the derivation of the formula (3.4) we have made use of the values of the following integrals:

$$\begin{aligned}\int_r^\alpha \frac{\operatorname{sh} {}^1/_2 x\,dx}{\sqrt{\operatorname{ch} x - \operatorname{ch} r}} = \sqrt{2}\ln\frac{2\sqrt{(\operatorname{ch}\alpha - \operatorname{ch} r)(1+\operatorname{ch}\alpha)} + 2\operatorname{ch}\alpha - \operatorname{ch} r + 1}{1+\operatorname{ch} r} \quad (\alpha > r) \\ \int_r^\alpha \frac{\operatorname{sh} {}^1/_2 x\,dx}{\operatorname{ch}^2 {}^1/_2 x\sqrt{\operatorname{ch} x - \operatorname{ch} r}} = \frac{2\sqrt{2}\sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}}{(1+\operatorname{ch} r)\sqrt{1+\operatorname{ch}\alpha}} \quad (\alpha > r) \\ \int_r^\alpha \frac{dx}{\operatorname{sh} x \operatorname{ch} {}^1/_2 x\sqrt{\operatorname{ch} x - \operatorname{ch} r}} = -\frac{\sqrt{2}\sqrt{\operatorname{ch}\alpha - \operatorname{ch} r}}{(1+\operatorname{ch} r)\sqrt{1+\operatorname{ch}\alpha}} + \\ + \frac{1}{2\sqrt{\operatorname{ch} r - 1}}\arccos\frac{4(1-\operatorname{ch} r) + (3-\operatorname{ch} r)(\operatorname{ch}\alpha - 1)}{(1+\operatorname{ch} r)(\operatorname{ch}\alpha - 1)} \quad (\alpha > r)\end{aligned} \tag{3.6}$$

In Table 2 we give the values of the quantity $\tau(r)\,(G\varepsilon)^{-1}$, obtained from the formula (3.4). In the second line of the Table we give for comparison the exact data in the problem of the torsion of an elastic semispace by a punch.

Then the values of the quantity $c\,(a\varepsilon)^{-1}$ are obtained for $\beta = \pi$ by the formula (3.5) (second row) with the use of the more exact approximation (2.10) (third row)

$b/a =$	0.1	0.3	0.5	0.7	0.9
$c\,(a\varepsilon)^{-1} =$	0.990	0.919	0.788	0 597	0.318
$c\,(a\varepsilon)^{-1} =$	1.01	0.949	0.825	0.618	0.319

In (2.10) for $\beta = \pi$ we set

$$D = 0.424, \qquad P_1(\gamma) = \gamma^2 + (0.75)^2, \qquad P_2(\gamma) = \gamma^2 + (0.56)^2$$

The error of such an approximation does not exceed 1.5%. Because of its awkwardness we do not give the analytic expression for $c(a\varepsilon)^{-1}$ which has been used for the given approximations.

In conclusion we note that in a similar manner we can investigate analytically and numerically the problem of the indentation of an annular punch into an elastic semispace and the problem of an annular crack in an elastic space.

Table 1

λ	0.4	0.6	0.8	1.0	1.2	1.4
$\varphi(0)$	—	2.35	1.97	1.76	1.62	1.52
$\varphi(1)$	—	1.53	1.60	1.53	1.46	1.41
P	—	4.16	3.70	3.36	3.13	2.96
$\varphi(0)$	3.06	2.33	1.96	1.75	1.62	1.52
$\varphi(1)$	2.29	1.89	1.68	1.56	1.47	1.41
P	5.60	4.36	3.74	3.37	3.14	2.97
$\varphi(0)$	3.06	2.26	1.87	1.64	1.49	1.39
$\varphi(1)$	2.08	1.75	1.55	1.42	1.33	1.27
P	5.50	4.19	3.53	3.14	2.88	2.69

Table 2

	b/a \ ρ/a	0.1	0.3	0.5	0.7	0.9
$\gamma=\pi$	0.0	0.128	0.400	0.735	1.25	2.63
	0.1	0.133	0.402	0.742	1.25	2.65
	0.3	0.132	0.411	0.753	1.28	2.68
	0.5	0.139	0.432	0.791	1.34	2.80
	0.7	0.149	0.465	0.857	1.46	3.07
	0.9	0.164	0.512	0.959	1.69	3.88
$\gamma={}^1/_2\pi$	0.1	0.137	0.404	0.743	1.26	2.66
	0.3	0.136	0.420	0.768	1.30	2.71
	0.5	0.147	0.456	0.832	1.40	2.90
	0.7	0.165	0.513	0.944	1.60	3.34
	0.9	0.192	0.597	1.13	2.01	4.65

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